Grade 11/12 Math Circles
November 30, 2022
Generating Functions 2

This week

Last week, we started talking about generating functions, a key tool in the field of Combinatorics. Today, we'll remind ourselves of what we learned last week, solve the Coin Problem and the Sicherman Dice Problem and finally talk a bit about integer compositions, a combinatorial class that is well-described by generating functions.

Recall

To begin, let's recall some definitions from last week.

Definition 1. A **set** is a collection of objects.

Examples: Sets

- $\{a, b, c\}$
- Set of 0-1 strings $= \{\varnothing, 0, 1, 00, 10, 01, 11, 000, \ldots\}$

Definition 2. A **weight function** is a mapping from a set to $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Example: 0-1 Strings

Define a weight function, w, as the length of the string. For example, $w(010) = 3$.

Definition 3. A **combinatorial class** is a set, paired with a weight function, such that there are a finite number of objects of any given weight.

Example: Combinatorial Class

The set of 0-1 strings, with the weight function $w(x) =$ length of the string, is a combinatorial class.

Exercise: Rolling a regular 8-sided die

Create a combinatorial class (including a set and weight function) to represent rolling a regular 8-sided die.

Exercise: Choosing 5 cent coins

Create a combinatorial class (including a set and weight function) to represent choosing some number of 5 cent coins from an infinitely large pile.

Definition 4. A **generating function** is a structure of the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

which stores information about combinatorial classes. In particular, $a_n =$ the number of objects of weight n.

Example: 0-1 Strings of Length Two

Find the generating function for the combinatorial class of 0-1 strings of length two with a weight function of the number of 1's in the string.

The four 0-1 strings of length two are 00, 01, 10 and 11. One string has no 1's, two have one 1 and one has two 1's. Thus, the generating function is $1 + 2z + z^2$.

Exercise: Rolling two regular 4-sided dice - Method 1

Find the generating function for the combinatorial class representing one roll of two regular 4-sided dice, where the weight function is the sum of the values rolled. To find this generating function, create a chart of the potential results from rolling the two dice.

Definition 5. Multiplication of combinatorial classes, $\mathcal{A} * \mathcal{B}$, is defined to give the following combinatorial class: the **set** is the objects of the form (a, b) where a is in the set of $\mathcal{A}$ and b is in the set of $\mathcal{B}$. The **weight function** is $w((a, b)) = w_A(a) + w_B(b)$. The **generating function** $A(z)B(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$. Note that the generating function is exactly as we would expect if we were simply multiplying the two polynomials.

Exercise: Rolling two regular 4-sided dice - Method 2

Find the generating function for the combinatorial class representing one roll of two regular 4-sided dice, where the weight function is the sum of the values rolled. This time, try using the multiplication of two combinatorial classes.

The Coin Problem

Now that we have reviewed the basics of generating functions, we can return to our motivating example of the Coin Problem:

How many ways can we make x cents using 5, 10 and 25 cent coins?

Recall that last week, we noted that listing all possibilities did not work well.

To answer this question with generating functions, we will create three combinatorial classes, one for each of the coin options, representing choosing how many coins of that type to take. Since we want the sum of the coin values to be a certain amount, we will use weight functions representing the value of the coins chosen. For this, we get:

$\mathcal{A}$:
Set:
Choices for number of 5¢ coins
WF: $w_A(x) = 5x$
GF:

$$\begin{aligned} A(z) &= z^0 + z^5 + z^{10} + \dots \\ &= \sum_{n=0}^{\infty} z^{5n} \\ &= \frac{1}{1-z^5} \end{aligned}$$

$\mathcal{B}$:
Set:
Choices for number of 10¢ coins
WF: $w_B(x) = 10x$
GF:

$$\begin{aligned} B(z) &= z^0 + z^{10} + z^{20} + \dots \\ &= \sum_{n=0}^{\infty} z^{10n} \\ &= \frac{1}{1-z^{10}} \end{aligned}$$

$\mathcal{C}$:
Set:
Choices for number of 25¢ coins
WF: $w_C(x) = 25x$
GF:

$$\begin{aligned} C(z) &= z^0 + z^{25} + z^{50} + \dots \\ &= \sum_{n=0}^{\infty} z^{25n} \\ &= \frac{1}{1-z^{25}} \end{aligned}$$

Then,
$\mathcal{A} * \mathcal{B} * \mathcal{C}$:
Set: (a, b, c), where a is from $\mathcal{A}$, b is from $\mathcal{B}$ and c is from $\mathcal{C}$. In other words, each element represents a choice of coins.
Weight Function: $w((a, b, c)) = w_A(a) + w_B(b) + w_C(c) = 5a + 10b + 25c$, in other words, the total value of the coins chosen.
Generating Function: $A(z)B(z)C(z) = \frac{1}{(1-z^5)(1-z^{10})(1-z^{25})}$

Sicherman Dice

Find two 6-sided dice such that:

- *Each side has a positive integer number of dots*
- *The two dice are not the same*
- *The probabilities of rolling a sum of* $2, \dots, 12$ *on these dice is the same as the probabilities for regular 6-sided dice*

Recall that, for the probabilities that we roll each of $2, \dots, 12$ to be the same, we need to have the same number of ways to roll each of the numbers $2, \dots, 12$.

Let's start by converting each of the requirements into generating function terminology. In particular, we want to represent the two new dice rolls as a generating functions.

1. Having a positive integer number of dots means that the new dice cannot have a constant (z^0) term.
2. To have different dice, the corresponding generating functions for each die must be different.
3. To have the same probabilities, the generating function created by multiplying the two dice generating functions together must be the same as for regular dice.
4. To be 6-sided, $A(1) = B(1) = 6$.

Last week, we found that the generating function of multiplying two regular dice is

$$F(z) = (z + z^2 + z^3 + z^4 + z^5 + z^6)^2$$

If we factor this polynomial, we get $F(z) = z^2(z+1)^2(z^2+z+1)^2(z^2-z+1)^2$.

In order to satisfy the first requirement, we need each die to have a factor of z. Since we need the dice to be 6-sided and $A(z) = C(z)D(z) \implies A(1) = C(1)D(1)$, we need a factor of $z+1$ and z^2+z+1 for each die. Finally, since the dice cannot be the same, this gives new dice with generating functions

$$A(z) = z(z+1)(z^2+z+1) = z + 2z^2 + 2z^3 + z^4$$

and

$$B(z) = z(z+1)(z^2+z+1)(z^2-z+1)^2 = z + z^2 + z^4 + z^5 + z^6 + z^8$$

So, our two new dice have sides $\{1, 2, 2, 3, 3, 4\}$ and $\{1, 3, 4, 5, 6, 8\}$.

Stop and Think

How can we check that our new dice indeed satisfy the requirements of the problem?

Integer Compositions

Let's now consider a combinatorial class which makes use of the generating function operations that we have learned.

Definition 6. A composition of n (technically an integer composition of size n) is an ordered tuple of positive integers which sum to n.

Example: Compositions of 3

The compositions of $n = 3$ are: (3), (2, 1), (1, 2) and (1,1,1). Note that (3) has one part, (2,1) and (1,2) have two parts and (1,1,1) has three parts.

Exercise: Compositions of 4

What are the compositions of 4?

Definition 7. The number of ways of choosing k items from n objects is

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k-1)}{k(k-1)\cdots 2\cdot 1} = \frac{n!}{k!(n-k)!}$$

where $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$.

Example: Choosing Socks

How many ways can we choose 2 socks from 4 different socks?

To represent that each sock is different, say that the socks are four colours: red (R), green (G), blue (B) and purple (P). To choose two socks, we could choose: RG, RB, RP, GB, GP or BP. So, we could choose 6 different ways.

$\binom{4}{2}$ represents the number of ways of making such a choice and we note that $\binom{4}{2} = \frac{4!}{2!2!} = \frac{4\cdot 3\cdot 2}{2\cdot 2} = 6$.

The **Negative Binomial Series** gives that:

$$\frac{1}{(1-z)^k} = \sum_{m=0}^{\infty} \binom{m+k-1}{k-1} z^m$$

Example: Compositions with k parts

How many compositions of n have k parts?

We begin by defining the following combinatorial class:

$\mathcal{A}$:

Set: $\{1, 2, 3, \ldots\}$

Weight Function: $w(x) = x$

Generating Function: $A(z) = z + z^2 + z^3 + \ldots = \frac{z}{1-z}$

We note that $\mathcal{A}$ defines the combinatorial class for the compositions of n with 1 part, where $[z^n]A(z) =$ the number of compositions of size n with 1 part.

Then,

$\mathcal{A}^k$:

Set: $(a_1, a_2, \ldots, a_k)$, where each a_i is an element of $\{1, 2, 3, \ldots\}$

Weight Function: $w((a_1, a_2, \ldots, a_k)) = a_1 + a_2 + \ldots + a_k$

Generating Function: $(A(z))^k = \frac{z^k}{(1-z)^k}$

This then gives the generating function for the compositions with k parts, where the weight function is the sum of the integers in the composition. So, we want the coefficient of the z^n term:

$$\begin{aligned}
[z^n]\frac{z^k}{(1-z)^k} &= [z^n]z^k\frac{1}{(1-z)^k} \\
&= [z^{n-k}]\frac{1}{(1-z)^k} \\
&= [z^{n-k}]\sum_{m=0}^{\infty}\binom{m+k-1}{k-1}z^m \\
&= \binom{n-k+k-1}{k-1} \\
&= \binom{n-1}{k-1}
\end{aligned}$$

Example: Compositions with k parts, each part even

How many compositions of n have k parts, where each part is an even number?

We begin by defining the following combinatorial class:
$\mathcal{A}$:
Set: $\{2, 4, 6, \ldots\}$
Weight Function: $w(x) = x$
Generating Function: $A(z) = z^2 + z^4 + z^6 + \ldots = \frac{z^2}{1-z^2}$

We note that $\mathcal{A}$ defines the combinatorial class for the compositions of n with 1 part, where that part is an even number.

Then,
$\mathcal{A}^k$:
Set: All $(a_1, a_2, \ldots, a_k)$, where each a_i is an element of $\{2, 4, 6, \ldots\}$
Weight Function: $w((a_1, a_2, \ldots, a_k)) = a_1 + a_2 + \ldots + a_k$
Generating Function: $(A(z))^k = \frac{z^{2k}}{(1-z^2)^k}$

This then gives the generating function for the compositions with k parts, with each part an even number, where the weight function is the sum of the integers in the composition. So, we want:

$$\begin{aligned}
[z^n]\frac{z^{2k}}{(1-z^2)^k} &= [z^n]z^{2k}\frac{1}{(1-z^2)^k} \\
&= [z^{n-2k}]\frac{1}{(1-z^2)^k} \\
&= [z^{n-2k}]\sum_{m=0}^{\infty}\binom{m+k-1}{k-1}z^{2m} \\
&= \begin{cases} \binom{\frac{n}{2}-1}{k-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$